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q-convolution and its q-Fourier transform

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Abstract

The functions on the lattice generated by the integer degrees of q^2 are considered, $0 < q < 1$. The q^2-translation operator is defined. The multiplicators and the q^2-convolutors are defined in the functional spaces which are dual with respect to the q^2-Fourier transform. The q^2-analog of convolution of two q^2-distributions is constructed. The q^2-analog of an arbitrary order derivative is introduced

1 Introduction

The Fourier transform plays an important part in the harmonic analysis on the simple Lie groups and on the homogeneous spaces. The concept of the convolution is closely connected with the Fourier transform because the last moves the convolution of two functions to the product of their images. In case of the quantum groups and quantum homogeneous spaces the q-analogs of the Fourier transform play the same part and the problem to construct the q-analog of the convolution arises. The different q-analogs of the Fourier transform have been investigated in [1, 2, 3, 4]. The q-convolution was introduced for the first time in [3]. It is extensive investigated in [5, 6]. The definition of the q-convolution is connected with the definition of the q-Fourier transform because the q-Fourier transform moves the q-convolution of two functions to the product of their images. The q-convolution considered in [5, 6] is connected with the q-Fourier transform considered in [4]. In these works the braided line is introduced.

In [7] the q^2-Fourier transform and the inversion formula have been constructed and they are quite similar to the classical ones [8, 9]. This construction coincides with the classical Fourier transform if $q \to 1$. The construction of the q^2-convolution corresponding the q^2-Fourier transform in the space of q^2-distributions [7] is proposed in the present work. Thus constructing of the theory similar to the classical one [8, 9] is prolonged. Moreover the braided line construction presents implicitly in this work, because we consider non commuting variables.

The q^2-convolution operator allows to determine the q^2-derivative of an arbitrary order.

In this paper we will use the same notation that in [7].

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2 Some preliminary relations

We assume that $z \in \mathbf{C}$ and $|q| < 1$, unless otherwise is specified.

We recall some notations [10]. For an arbitrary a

$$(a,q)_n = \begin{cases} 1 & \text{for} \quad n = 0 \\ (1-a)(1-aq)\dots(1-aq^{n-1}) & \text{for} \quad n \geq 1, \end{cases}$$

$$(a,q)_\infty = \lim_{n\to\infty}(a,q)_n,$$

$$\left[\begin{array}{c} l \\ i \end{array}\right]_{q^2} = \frac{(q^2,q^2)_l}{(q^2,q^2)_i(q^2,q^2)_{l-i}}.$$

Consider the q^2-exponentials

$$e_{q^2}(z) = \sum_{n=0}^{\infty} \frac{z^n}{(q^2,q^2)_n} = \frac{1}{(z,q^2)_\infty}, \qquad |z| < 1, \tag{2.1}$$

$$E_{q^2}(z) = \sum_{n=0}^{\infty} \frac{q^{n(n-1)} z^n}{(q^2,q^2)_n} = (-z,q^2)_\infty \tag{2.2}$$

and the basic hypergeometric series

$$ {}_r\Phi_s(a_1,\dots,a_r;b_1,\dots,b_s;q^2,z) =$$

$$= \sum_{n=0}^{\infty} \frac{(a_1,q^2)_n \cdots (a_r,q^2)_n}{(q^2,q^2)_n(b_1,q^2)_n \cdots (b_s,q^2)_n} [(-1)^n q^{\binom{n}{2}}]^{1+s-r} z^n. \tag{2.3}$$

We consider the series

$$\mathbf{Q}(z,q) = (1-q^2) \sum_{m=-\infty}^{\infty} \frac{1}{zq^{2m} + z^{-1}q^{-2m}}$$

expressed by theta-function (see [7]). Assume

$$\Theta_0 = \mathbf{Q}(1-q^2,q). \tag{2.4}$$

Let $\mathcal{A} = C(z,z^{-1})$ be the algebra of formal Laurent series. The q^2-derivative of $f(z) \in \mathcal{A}$ is defined as

$$\partial_z f(z) = \frac{z^{-1}}{1-q^2}(f(z) - f(q^2 z)).$$

For an arbitrary $n \geq 0$

$$\partial_z^k z^n = \begin{cases} \frac{(q^2,q^2)_n}{(q^2,q^2)_{n-k}(1-q^2)^k} z^{n-k} & \text{for} \quad 0 \leq k \leq n \\ 0 & \text{for} \quad k > n, \end{cases} \tag{2.5}$$

and for any $n \geq 0$ and $k \geq 0$

$$\partial_z^k z^{-n-1} = (-1)^k q^{-k(2n+k+1)} \frac{(q^2,q^2)_{n+k}}{(q^2,q^2)_n(1-q^2)^k} z^{-n-k-1}. \tag{2.6}$$

The q^2-integral (Jackson integral [10]) is defined as the map I_{q^2} from $\mathcal{A}$ to the space of formal number series

$$I_{q^2} f = \int d_{q^2} z f(z) = (1-q^2) \sum_{m=-\infty}^{\infty} q^{2m} [f(q^{2m}) + f(-q^{2m})]$$

Definition 2.1 *$f(z)$ is absolutely q^2-integrable function, if the series*

$$\sum_{m=-\infty}^{\infty} q^{2m} [|f(q^{2m})| + |f(-q^{2m})|]$$

converges.

Let $\mathcal{B}$ be the algebra analogous to $\mathcal{A}$, but generated by s, s^{-1} which commute with z as $zs = q^2 sz$.

We denote by $\mathcal{AB}$ the whole algebra with generators z, z^{-1}, s, s^{-1} and relations

$$zs = q^2 sz, \quad \partial_z s = q^{-2} s \partial_z, \quad \partial_s z = q^2 z \partial_s, \quad \partial_z \partial_s = q^2 \partial_s \partial_z. \tag{2.7}$$

We will consider $\mathcal{AB}$ as a left module under the action of $\mathcal{A}$ by multiplication, and a right module under the action of $\mathcal{B}$.

To define the q^2-integral on $\mathcal{AB}$ we order the generators of integrand in such a way that z stays on the left side while s stays on the right side. For example, if $f(z) = \sum_r a_r z^r$ then

$$f(zs) = \sum_r a_r (zs)^r = \sum_r a_r q^{-r(r-1)} z^r s^r.$$

For convenience we introduce the following notation

$$\ddagger g(zs) \ddagger = \sum_r a_r z^r s^r, \text{ if } g(z) = \sum_r a_r z^r.$$

For example, we can derive from (2.1), (2.2) and (2.7) that

$$E_{q^2}((1-q^2)zs) = \ddagger e_{q^2}((1-q^2)zs) \ddagger.$$

We define the operator

$$\Lambda_z : f(z) \to f(q^2 z).$$

Obviously

$$\Lambda_z z = q^2 z \Lambda_z, \qquad \partial_z \Lambda_z = q^2 \Lambda_z \partial_z.$$

3 q^2-distributions and the q^2-Fourier transform

In [7] we have defined the spaces of the test functions and the q^2-distributions and the q^2-Fourier transform has been constructed. We reproduce some statements here.

Let $S_{q^2} = \{\phi(x)\}$ be the space of infinitely q^2-differentiable fast decreasing functions

$$|x^k \partial_x^l \phi(x)| \le C_{k,l}(q), \; k \ge 0, \; l \ge 0. \tag{3.1}$$

Let S be the space of infinitely differentiable (in the classic sense) fast decreasing function

$$|x^k\phi^{(l)}(x)| \leq C_{k,l},\ \ k \geq 0\ l \geq 0.$$

It has been show in [7] that $S \subset S_{q^2}$. In addition

$$\partial_x^l \phi(x)|_{x=0} = \frac{(q^2,q^2)_l}{(1-q^2)^l l!}\phi^{(l)}(0). \tag{3.2}$$

Proposition 3.1 *If $\phi(z) \in S_{q^2}$, then*

$$\int d_{q^2}z\partial_z\phi(z) = 0.$$

Corollary 3.1 *(q^2-integration by parts). For any $k \geq 0$*

$$\int d_{q^2}z\phi(z)\partial_z^k\psi(z) = (-1)^k q^{-k(k-1)}\int d_{q^2}z\partial_z^k\phi(z)\psi(q^{2k}z). \tag{3.3}$$

Definition 3.1 *The q^2-distribution f over S_{q^2} is a linear continuous functional*

$$f \rightarrow < f,\phi >,\quad \phi(z) \in S_{q^2}.$$

We denote by S'_{q^2} the space of the q^2-distributions over S_{q^2}.

The q^2-distributions defined by the q^2-integral

$$< f,\phi >= \int_{-\infty}^{\infty} d_{q^2}z\overline{f(z)}\phi(z) = (1-q^2)\sum_{m=-\infty}^{\infty} q^{2m}[\overline{f(q^{2m})}\phi(q^{2m}) + \overline{f(-q^{2m})}\phi(-q^{2m})]$$

we refer as a regular one.

Proposition 3.1 and Corollary 3.1 allow to introduce the q^2-differentiation in S'_{q^2}

$$< \partial_z f,\phi >= - < \Lambda_z f,\partial_z\phi > . \tag{3.4}$$

It follows from (3.4) that the conjugate operator for ∂_z^k for any $k \geq 0$ has the form

$$(\partial_z^k)^* = (-1)^k q^{k(k-1)}\partial_z^k\Lambda_z^{-k}. \tag{3.5}$$

The change of variables $q^{-2k}z \rightarrow z$ in the q^2-integral leads to

$$(\Lambda_z^{-k})^* = q^{2k}\Lambda_z^k. \tag{3.6}$$

Definition 3.2 *f is the q^2-distribution with multiplicity p of the q^2-singularity if it is represented in form*

$$f = \sum_{k=0}^{p}\partial_z^k f_k(z),$$

where $f_k(z)$ are the ordinary functions growing no faster then some power of $|z|$ as $|z| \rightarrow \infty$.

For example $\delta_{q^2}(z)$ is the q^2-distribution of multiplicity one of the q^2-singularity because for an arbitrary $\phi(z) \in S_{q^2}$

$$< \delta_{q^2}, \phi >=< \frac{1}{2}\partial_z(\theta^+_{q^2} - \theta^-_{q^2}), \phi >= - < \frac{1}{2}\Lambda_z(\theta^+_{q^2} - \theta^-_{q^2}), \partial_z\phi >== -\frac{1}{2}\int_0^\infty d_{q^2}z\partial_z\phi(z)+$$

$$+\frac{1}{2}\int_{-\infty}^0 d_{q^2}z\partial_z\phi(z) = -\frac{1}{2}\sum_{m=-\infty}^{\infty}[\phi(q^{2m}) - \phi(q^{2m+2}) + \phi(-q^{2m}) - \phi(-q^{2m+2})] =$$

$$= \lim_{m\to\infty}\frac{\phi(q^{2m}) + \phi(-q^{2m})}{2} = \phi(0).$$

Let the space $S^{q^2} = \{\psi(s)\}$ be the copy of the $S_{q^2} = \{\phi(z)\}$ (3.1), but s and z behave as the generators of the algebra $\mathcal{AB}$ (2.7). Introduce the same topology in S^{q^2} as one in S_{q^2}

$$|s^k\partial^l_s\phi(s)| \leq C_{k,l}(q),\ \ k \geq 0,\ \ l \geq 0,$$

Thereby these spaces are isomorphic.

The q^2-Fourier transform $\mathcal{F}_{q^2}$, i.e. the map S_{q^2} into S^{q^2} has been constructed in [7]

$$S_{q^2} \xrightarrow{\mathcal{F}_{q^2}} S^{q^2}$$

where

$$\mathcal{F}_{q^2}\phi(z) = \int d_{q^2}z\phi(z)\ {}_0\Phi_1(-;0;q^2, i(1-q^2)q^2zs),$$

and ${}_0\Phi_1$ is determined by (2.3). The inverse transform

$$\mathcal{F}^{-1}_{q^2}\psi(s) = \frac{1}{2\Theta_0}\int E_{q^2}(-i(1-q^2)zs)\psi(s)d_{q^2}s, \quad \psi(s) \in S^{q^2}, \tag{3.7}$$

has been constructed and their continuity was proofed. The constant Θ_0 is determined by (2.4).

The following relations are valid

$$\mathcal{F}_{q^2}\Lambda_z = q^{-2}\Lambda_s^{-1}\mathcal{F}_{q^2}, \qquad \mathcal{F}_{q^2}\partial_z = -is\mathcal{F}_{q^2}, \qquad \mathcal{F}_{q^2}z = -iq^{-2}\Lambda_s^{-1}\partial_s\mathcal{F}_{q^2},$$

$$\mathcal{F}^{-1}_{q^2}\Lambda_s = q^{-2}\Lambda_z^{-1}\mathcal{F}^{-1}_{q^2}, \qquad \mathcal{F}^{-1}_{q^2}\partial_s = i\Lambda_z^{-1}z\mathcal{F}^{-1}_{q^2}, \qquad \mathcal{F}^{-1}_{q^2}s = i\partial_z\mathcal{F}^{-1}_{q^2}. \tag{3.8}$$

Definition 3.3 *The q^2-Fourier transform of a q^2-distribution $f \in S'_{q^2}$ is the q^2-distribution $g \in (S^{q^2})'$ defined by the equality*

$$< g, \psi >=< f, \phi >, \quad \psi(s) = \mathcal{F}_{q^2}\phi(z), \tag{3.9}$$

where $\phi(z)$ is an arbitrary function from S_{q^2}.

Presuppose that the q^2-distribution f corresponds to $f(z)$ and $zf(z)$ is absolutely q^2-integrable function. Let $\phi(z) = \mathcal{F}_{q^2}^{-1}\psi(s)$. Then

$$< f, \phi >= \frac{1}{2\Theta_0}\int d_{q^2}z\overline{f(z)}\int E_{q^2}(-i(1-q^2)zs)\psi(s)d_{q^2}s =$$

$$= \frac{1}{2\Theta_0}\int\overline{\int d_{q^2}zf(z)E_{q^2}(i(1-q^2)zs)}\psi(s)d_{q^2}s =< g, \psi > .$$

It means that the q^2-distribution g corresponds to the function

$$g(s) = \frac{1}{2\Theta_0}\int d_{q^2}zf(z)E_{q^2}(i(1-q^2)zs). \tag{3.10}$$

In the same way, if g is determined by the absolutely q^2-integrable function $g(s)$ and $\psi(s) = \mathcal{F}_{q^2}\phi(z)$, then

$$< g, \psi >= \int\int d_{q^2}z\phi(z)\ {}_0\Phi_1(-;0;q^2, i(1-q^2)q^2zs)\overline{g(s)}d_{q^2}s =$$

$$= \int d_{q^2}z\phi(z)\overline{\int\ {}_0\Phi_1(-;0;q^2, -i(1-q^2)q^2zs)g(s)d_{q^2}s} =< f, \phi >,$$

i.e. f corresponds to

$$f(z) = \int\ {}_0\Phi_1(-;0;q^2, -i(1-q^2)q^2zs)g(s)d_{q^2}s. \tag{3.11}$$

The q^2-Fourier transform of a q^2-distribution from S'_{q^2} we denote by $\mathcal{F}'_{q^2}$. It follows from (3.10), (3.11) and (3.8) that in the space of q^2-distributions the following commutative relations are valid

$$\mathcal{F}'_{q^2}\Lambda_z = q^{-2}\Lambda_s^{-1}\mathcal{F}'_{q^2}, \qquad \mathcal{F}'_{q^2}\partial_z = -i\Lambda_s^{-1}s\mathcal{F}'_{q^2}, \qquad \mathcal{F}'_{q^2}z = -i\partial_s\mathcal{F}'_{q^2},$$

$$(\mathcal{F}'_{q^2})^{-1}\Lambda_s = q^{-2}\Lambda_z^{-1}(\mathcal{F}'_{q^2})^{-1}, (\mathcal{F}'_{q^2})^{-1}\partial_s = iz(\mathcal{F}'_{q^2})^{-1}, (\mathcal{F}'_{q^2})^{-1}s = iq^{-2}\Lambda_z^{-1}\partial_z(\mathcal{F}'_{q^2})^{-1}. \tag{3.12}$$

4 q^2-shift in the space of the test functions

Let ξ be an element of the same nature as s so their sum is determined

$$s + \xi = \xi + s,$$

and the commutative relations

$$\xi s = q^2 s\xi, \quad \xi\partial_s = q^{-2}\partial_s\xi, \quad \Lambda_s\xi = \xi\Lambda_s \tag{4.1}$$

are fulfilled. In this case we will call the element s subordinate to ξ.

Definition 4.1 *We will call the operator*

$$T_\xi = e_{q^2}((1-q^2)\xi\Lambda_s^{-1}\partial_s) \tag{4.2}$$

by q^2-shift in the space S^{q^2}.

Proposition 4.1 *For an arbitrary function $g(s) \in \mathcal{B}$*

$$T_\xi g(s) = g(s+\xi).$$

Proof. It follows from (2.1), (4.1) and (2.6) that for an arbitrary $n \geq 0$

$$e_{q^2}((1-q^2)\xi\Lambda_s^{-1}\partial_s)s^n = \ddagger e_{q^2}((1-q^2)\xi\Lambda_s^{-1}\partial_s)\ddagger s^n = \sum_{k=0}^{n} \left[\begin{array}{c} n \\ k \end{array}\right]_{q^2} s^{n-k}\xi^k = (s+\xi)^n. \tag{4.3}$$

By induction on n for any $n \geq 0$ one finds

$$(s+\xi)^{-n-1} = \sum_{k=0}^{\infty}(-1)^k \left[\begin{array}{c} n+k \\ k \end{array}\right]_{q^2} s^{-n-k-1}\xi^k.$$

Therefore, it follows from (2.1), (4.1) and (2.7) for any $n \geq 0$

$$e_{q^2}((1-q^2)\xi\Lambda_s^{-1}\partial_s)s^{-n-1} = \ddagger e_{q^2}((1-q^2)\xi\Lambda_s^{-1}\partial_s)\ddagger s^{-n-1} = (s+\xi)^{-n-1}. \tag{4.4}$$

Now it follows from (4.3) and (4.4) that for $g(s) = \sum_r a_r s^r$

$$e_{q^2}((1-q^2)\xi\Lambda_s^{-1}\partial_s)g(s) = g(s+\xi).$$

■

Proposition 4.2 *If ξ_2 is subordinated ξ_1, i.e.*

$$\xi_1\xi_2 = q^2\xi_2\xi_1,$$

$$T_{\xi_2}T_{\xi_1} = T_{\xi_1+\xi_2}.$$

Proof. It follows from (2.1) and (5.1)

$$e_{q^2}((1-q^2)\xi_2\Lambda_s^{-1}\partial_s)e_{q^2}((1-q^2)\xi_1\Lambda_s^{-1}\partial_s) = \sum_{k=0}^{\infty}\frac{(1-q^2)^k}{(q^2,q^2)_k}\xi_2^k\Lambda_s^{-k}\partial_s^k\sum_{l=0}^{\infty}\frac{(1-q^2)^l}{(q^2,q^2)_l}\xi_1^l\Lambda_s^{-l}\partial_s^l =$$

$$= \sum_{m=0}^{\infty}\frac{(1-q^2)^m}{(q^2,q^2)_m}\sum_{l=0}^{m}\left[\begin{array}{c} m \\ l \end{array}\right]_{q^2}\xi_2^{m-l}\Lambda_s^{-m+l}\partial_s^{m-l}\xi_1^l\Lambda_s^{-l}\partial_s^l =$$

$$= \sum_{m=0}^{\infty}\frac{(1-q^2)^m}{(q^2,q^2)_m}\sum_{l=0}^{m}\left[\begin{array}{c} m \\ l \end{array}\right]_{q^2}\xi_2^{m-l}\xi_1^l\Lambda_s^{-m}\partial_s^m =$$

$$= \sum_{m=0}^{\infty}\frac{(1-q^2)^m}{(q^2,q^2)_m}(\xi_1+\xi_2)^m\Lambda_s^{-m}\partial_z^m = e_{q^2}((1-q^2)(\xi_1+\xi_2)\Lambda_s^{-1}\partial_z).$$

■

Rule (O)(Order). *If a function depends on several variables then it is necessary to put them in order according to subordination before one takes its restriction on the lattice $\{q^{2n}\}$ so that the subordinate variable stands to the right.*

Proposition 4.3 *Operator T_ξ for $\xi = q^{2m}$ can be represented by the form*

$$T_{q^{2m}} = \sum_{k=0}^{\infty} \frac{q^{2k(k+m)}}{(q^2,q^2)_k} s^{-k} E_{q^2}(-q^{2(m+k+1)}s^{-1})\Lambda_s^{-k}. \tag{4.5}$$

Proof. It is easily to prove by the induction on k that for an arbitrary $k \geq 0$

$$\partial_s^k \psi(s) = \frac{1}{(1-q^2)^k s^k} \sum_{l=0}^{k} (-1)^l q^{-l(2k-l-1)} \left[\begin{array}{c} k \\ l \end{array} \right]_{q^2} \psi(q^{2l}s).$$

It follows from (4.1) and (4.2) that

$$T_\xi \psi(s) = \sum_{k=0}^{\infty} \xi^k q^{2k^2} s^{-k} \sum_{l=0}^{k} (-1)^l \frac{q^{-l(2k-l-1)}}{(q^2,q^2)_l (q^2,q^2)_{k-l}} \psi(q^{2l-2k}s) =$$

$$= \sum_{l=0}^{\infty} \frac{(-1)^l q^{l(l+1)}}{(q^2,q^2)_l} \sum_{k=0}^{\infty} \frac{q^{2k(k+l)}}{(q^2,q^2)_k} \xi^{k+l} s^{-k-l} \psi(q^{-2k}s) =$$

$$\sum_{k=0}^{\infty} \frac{q^{2k^2}}{(q^2,q^2)_k} \xi^k \left(\sum_{l=0}^{\infty} (-1)^l \frac{q^{l(l+2k+1)}}{(q^2,q^2)_k} \xi^l z^{-l} \right) s^{-k} \psi(q^{-2k}s).$$

It is possible to change the order of summation because the inner series are converged uniformly with respect to k and l. Substituting $\xi = q^{2m}$ we obtain

$$T_{q^{2m}} \psi(s) = \sum_{k=0}^{\infty} \frac{q^{2k(m+k)} s^{-k}}{(q^2,q^2)_l} \psi(q^{-2k}s) \sum_{l=0}^{\infty} (-1)^l \frac{q^{l(l-1)} q^{2l(m+k+1)} s^{-l}}{(q^2,q^2)_l} =$$

$$= \sum_{k=0}^{\infty} \frac{q^{2k(m+k)}}{(q^2,q^2)_l} s^{-k} E_{q^2}(-q^{2(m+k+1)}s^{-1}) \psi(q^{-2k}s). \tag{4.6}$$

■

Corollary 4.1 *If $\psi(s)$ is limited by a constant C i.e. for an arbitrary n $\quad |\psi(q^{2n})| \leq C$ then for any n and m*

$$|T_{q^{2m}} \psi(q^{2n})| \leq C.$$

Proof. It is seen from (2.2) that if $k > 0$ then $E_{q^2}(-q^{2k}) > 0$, and if $k \leq 0$ then $E_{q^2}(-q^{2k}) = 0$. Substituting $s = q^{2n}$ in (4.5) we obtain

$$|T_{q^{2m}} \psi(q^{2n})| \leq C \sum_{k=0}^{\infty} \frac{q^{2k(k+m-n)}}{(q^2,q^2)_k} E_{q^2}(q^{2(k+m+1-n)}) =$$

$$= C \sum_{k=0}^{\infty} \frac{q^{2k(k+m-n)}}{(q^2,q^2)_k} \sum_{l=0}^{\infty} \frac{(-1)^l q^{l(l+1)} q^{2l(k+m-n)}}{(q^2,q^2)_l} = C \sum_{k=0}^{\infty} \frac{q^{2k(k+m-n)}}{(q^2,q^2)_k} \sum_{l=0}^{k} (-1)^l \left[\begin{array}{c} k \\ l \end{array} \right]_{q^2} q^{-l(2k-l-1)}.$$

By induction on k

$$\sum_{l=0}^{k} (-1)^l \left[\begin{array}{c} k \\ l \end{array} \right]_{q^2} q^{-l(2k-l-1)} = \begin{cases} 1 & \text{for} \quad k = 0 \\ 0 & \text{for} \quad k \geq 1. \end{cases}$$

From here the statement of the Corollary follows.■

Proposition 4.4 *If $\psi(s) \in S^{q^2}$ then*

$$\int T_\xi \psi(s) d_{q^2} s = \int \psi(s) d_{q^2} s. \tag{4.7}$$

Proof. It follows from (4.2) that

$$T_\xi \psi(s) = \psi(s) + \sum_{k=1}^{\infty} \frac{(1-q^2)^k}{(q^2,q^2)_k} \xi^k \Lambda_s^{-k} \partial_s^k \psi(s).$$

(4.7) follows from Proposition 3.1. ∎

Proposition 4.5 *For $\psi(s) \in S^{q^2}$, for any $n \geq 0, \quad m \geq 0$ and for any $r, \quad t$*

$$|s^n \partial_s^m T_\xi \psi(s)|_{s=q^{2r}, \xi=q^{2t}}| \leq C_{n,m}.$$

Proof. By induction on k

$$\partial_s^k (s\psi(s)) = \frac{1-q^{2k}}{1-q^2} \partial_s^{k-1} \psi(s) + q^{2k} s \partial_s^k \psi(s).$$

We get from here and (4.2)

$$sT_\xi \psi(s) = T_{q^{-2}\xi}(s\psi(s)) - q^{-2} \xi \Lambda_s^{-1} T_{q^{-2}\xi} \psi(s).$$

By induction on n

$$s^n T_\xi \psi(s) = \sum_{l=0}^{n} (-1)^l C_n^l q^{-2l} \xi^l \Lambda_s^{-l} T_{q^{-2n}\xi}(\psi(s) s^{n-l}). \tag{4.8}$$

It follows from (4.1) that for any $m \geq 0$

$$\partial_s^m T_\xi \psi(s) = T_\xi \partial_s^m \psi(s). \tag{4.9}$$

Proposition follows from (4.8), (4.9) and from Corollary 4.1.∎

The next theorem follows from Proposition 4.5

Theorem 4.1 *The translation operator T_ξ is the bounded operator in the space S^{q^2}.*

Using (3.4), (3.5) we define the conjugate operator T_ξ^* in the space of the q^2-distributions $(S^{q^2})'$:

$$T_\xi^* = e_{q^2}(-(1-q^2) q^2 \xi \partial_s), \qquad \xi s = q^2 s \xi. \tag{4.10}$$

If $\xi_1 \xi_2 = q^2 \xi_2 \xi_1$ then

$$T_{\xi_2}^* T_{\xi_1}^* = T_{\xi_1 + \xi_2}^*.$$

5 q^2-convolution

Let $g(s)$ be the q^2-distribution determined on the space S^{q^2} of functions of one variable s, and $r(\xi)$ be the q^2-distribution determined on the space S^{q^2} of functions of one variable ξ, moreover s and ξ are connected by relation (4.1). We keep the designation S^{q^2} for the space of functions of two variables $\psi(\xi, s)$. Then the functional

$$r(\xi) \times g(s)$$

is well-defined on this space and we call it the direct product of the functionals $r(\xi)$ and $g(s)$

$$< g(s), < r(\xi), \psi(\xi, s) >>=< r(\xi), < g(s), \psi(\xi, s) >> .$$

In addition if the functionals r and g are regular then we must succeed to Rule (**O**).

Definition 5.1 *We will call the functional*

$$< r * g, \psi >=< r(\xi) \times g(s), T_\xi \psi(s) > \tag{5.1}$$

by the q^2-convolution of two q^2-distributions from $(S^{q^2})'$.

Definition 5.2 *$h(z)$ is the multiplicator in S_{q^2} if for an arbitrary function $\phi(z) \in S_{q^2}$ $\;h(z)\phi(z) \in S_{q^2}$.*

Proposition 5.1 *The multiplication on an infinitely q^2-differentiable function $h(z)$ complying with inequality*

$$|\partial_z^k h(z)| \le C_k(1 + |z|^l) \tag{5.2}$$

for some $l \ge 0$ is a bounded operator in S_{q^2}.

Proof. By induction on m

$$\partial_z^m[h(z)\phi(z)] = \sum_{i=0}^{m} \begin{bmatrix} m \\ i \end{bmatrix}_{q^2} q^{-2i(m-i)} \partial_z^{m-i} h(q^{2i}z) \partial_z^i \phi(z).$$

Then for any $n \ge 0, \quad m \ge 0$

$$|z^n \partial_z^m[h(z)\phi(z)]| = |z^n \sum_{i=0}^{m} \begin{bmatrix} m \\ i \end{bmatrix}_{q^2} q^{-2i(m-i)} \partial_z^{m-i} h(q^{2i}z) \partial_z^i \phi(z)| \le$$

$$\le \sum_{i=0}^{m} \begin{bmatrix} m \\ i \end{bmatrix}_{q^2} q^{-2i(m-i)} |\partial_z^{m-i} h(q^{2i}z)||z^m \partial_z^i \phi(z)|.$$

It follows from (3.1) and (5.2) that

$$|z^n \partial_z^m[u(z)\phi(z)]| \le \sum_{i=0}^{m} \begin{bmatrix} m \\ i \end{bmatrix}_{q^2} q^{-2i(m-i)} C_{m-i}(1 + |z|^l)|z^m \partial_z^i \phi(z)| \le$$

$$\le \sum_{i=0}^{m} \begin{bmatrix} m \\ i \end{bmatrix}_{q^2} q^{-2i(m-i)} C_{m-i} C_{m,i} + \sum_{i=0}^{m} \begin{bmatrix} m \\ i \end{bmatrix}_{q^2} q^{-2i(m-i)} C_{m-i} C_{m+l,i} = \tilde{C}_{m+l,n}.$$

■

Thus the functions satisfying (5.2), i.e. growing no faster then some power of $|z|$ for $|z| \to \infty$, are the multiplicator in S_{q^2}.

Definition 5.3 *We will call functional* $r \in (S^{q^2})'$ *by* q^2*-convolutor in* S^{q^2}*, if for an arbitrary function* $\psi(s) \in S^{q^2}$ *the* q^2*-convolution*

$$(r * \psi)(s) = \int d_{q^2}\xi \overline{r(\xi)} T_\xi \psi(s)$$

exists and belongs to S^{q^2}.

Proposition 5.2 *If functional* $h \in S_{q^2}{}'$ *is the multiplicator in* S_{q^2}*, then its* q^2*-Fourier transform* $\mathcal{F}'_{q^2} h = r \in (S^{q^2})'$ *is the* q^2*-convolutor in* S^{q^2}.

Proof. Let functional h correspond to the infinitely q^2-differentiable function $h(z)$, and $zh(z)$ be an absolutely q^2-integrable. Then functional $\mathcal{F}'_{q^2} h = r$ corresponds to the function $r(s)$ and for any k and $n \geq 0$ a constant $C_n > 0$ exists so that $q^{2kn}|r(\pm q^{2k})| \leq C_n$. It follows from here and from Theorem 4.1 that the series

$$(1-q^2)\sum_{k=-\infty}^{\infty} q^{2k}[\overline{r(q^{2k})}T_{q^{2k}}\psi(s) + \overline{r(-q^{2k})}T_{-q^{2k}}\psi(s)]$$

converges uniformly with respect to s together with q^2-derivatives with respect to s and we can q^2-differentiate it term-by-term.

It is easily to show as in the proof of Proposition 4.5 that for any $n \geq 0$ and $m \geq 0$

$$s^n \partial_s^m T_{\pm q^{2k}}\psi(s) = \sum_{l=0}^{n}(-1)^l \left[\begin{array}{c} n \\ l \end{array}\right]_{q^2} q^{2l(k-m)} \Lambda_s^{-l} T_{\pm q^{2(k-m)}}(s^{n-l}\psi(s)). \tag{5.3}$$

Hence, for any $n \geq 0$ and $m \geq 0$ and for an arbitrary $\psi(s) \in S^{q^2}$

$$|s^n \partial_s^m (r * \psi)(s)| \leq (1-q^2)\sum_{k=-\infty}^{\infty} q^{2k}[|\overline{r(q^{2k})}s^n \partial_s^m T_{q^{2k}}\psi(s)| + |\overline{r(-q^{2k})}s^n \partial_s^m T_{-q^{2k}}\psi(s)|].$$

Proposition follows from Corollary 4.1 and (5.3).

Let now functional h correspond to infinitely q-differentiable function $h(z)$ growing for $|z| \to \infty$ no faster then $|z|^p$, $p > 0$ is integer . That is $h(z)$ has form

$$h(z) = \sum_{k=0}^{p} z^k h_k(z),$$

where $h_k(z)$ are such infinitely q^2-differentiable functions that $zh_k(z)$ are absolutely q^2-integrable. It follows from (3.12) that

$$r(s) = \sum_{k=0}^{p}(-i\partial_s)^k r_k(s), \quad r_k(s) = \mathcal{F}'_{q^2} h_k(z).$$

On the other hand using the formula of q^2-integration by parts (3.2) it is easily to obtain

$$\int d_{q^2}\xi \overline{\partial_\xi^k r_k(\xi)} T_\xi \psi(s) = (-1)^k q^{k(k-1)} \int d_{q^2} \overline{r_k(s)} T_\xi(\partial_s^k \psi(q^{-2k}s)). \tag{5.4}$$

So the statement of the Proposition is truly in this case also. ∎

The formula

$$(\partial_x i^k r * \psi)(s) = (-1)^{-k(k+1)} \Lambda_s^{-k} \partial_s^k (r * \psi)(s). \tag{5.5}$$

follows from (5.4)

Proposition 5.3 *If r is the q^2-convolutor in S^{q^2}, then the q^2-convolution $(r*\psi)(s)$ is the q^2-Fourier transform of the product $\overline{h(z)}\phi(z)$, where $\phi(z)=\mathcal{F}_{q^2}^{-1}\psi(s)$ and $h(z)=(\mathcal{F}'_{q^2})^{-1}r(s)$.*

Proof. It was proved in [7] (Lemma (5.2)) that

$$\int E_{q^2}(-i(1-q^2)zs)E_{q^2}(i(1-q^2)q^2s)d_{q^2}s=\begin{cases}\frac{2}{1-q^2}\Theta_0 & \text{for} \quad z=1\\ 0 & \text{for} \quad z\neq 1.\end{cases} \tag{5.6}$$

It is easily to convince that

$$e_{q^2}((1-q^2)\xi\Lambda_s^{-1}\partial_s)E_{q^2}(i(1-q^2)q^{2n+2}s)=E_{q^2}(i(1-q^2)q^{2n+2}\xi)E_{q^2}(i(1-q^2)q^{2n+2}s). \tag{5.7}$$

At first let r be a regular functional corresponding to function $r(s)$. It follows from (3.7) and (3.10)

$$\int d_{q^2}\xi\overline{r(\xi)}e_{q^2}((1-q^2)\xi\Lambda_s^{-1}\partial_s)\psi(s)=$$

$$=\frac{(1-q^2)^2}{2\Theta_0}\int d_{q^2}\xi\sum_{m=-\infty}^{\infty}q^{2m}[\overline{h(q^{2m})}e_{q^2}(-i(1-q^2)q^{2m}\xi)+\overline{h(-q^{2m})}e_{q^2}(i(1-q^2)q^{2m}\xi)]\times$$

$$\times e_{q^2}((1-q^2)\xi\Lambda_s^{-1}\partial_s)\sum_{n=-\infty}^{\infty}q^{2n}[\phi(q^{2n})E_{q^2}(i(1-q^2)q^{2n+2}s)+\phi(-q^{2n})E_{q^2}(-i(1-q^2)q^{2n+2}s)].$$

These series converge uniformly with respect to ξ and so we can q^2-integrate them term-by-term. Then using (5.6) and (5.7), we obtain

$$\int d_{q^2}\xi\overline{r(\xi)}e_{q^2}((1-q^2)\xi\Lambda_s^{-1}\partial_s)\psi(s)=\frac{(1-q^2)^2}{2\Theta_0}\sum_{m=-\infty}^{\infty}q^{2m}\sum_{n=-\infty}^{\infty}q^{2n}\times$$

$$\times[\overline{h(q^{2m})}\phi(q^{2n})\int d_{q^2}\xi e_{q^2}(-i(1-q^2)q^{2m}\xi)E_{q^2}(i(1-q^2)q^{2n+2}\xi)E_{q^2}(i(1-q^2)q^{2n+2}s)+$$

$$+\overline{h(q^{2m})}\phi(-q^{2n})\int d_{q^2}\xi e_{q^2}(-i(1-q^2)q^{2m}\xi)E_{q^2}(-i(1-q^2)q^{2n+2}\xi)E_{q^2}(-i(1-q^2)q^{2n+2}s)+$$

$$+\overline{h(-q^{2m})}\phi(q^{2n})\int d_{q^2}\xi e_{q^2}(i(1-q^2)q^{2m}\xi)E_{q^2}(i(1-q^2)q^{2n+2}\xi)E_{q^2}(i(1-q^2)q^{2n+2}s)+$$

$$+\overline{h(-q^{2m})}\phi(-q^{2n})\int d_{q^2}\xi e_{q^2}(i(1-q^2)q^{2m}\xi)E_{q^2}(-i(1-q^2)q^{2n+2}\xi)E_{q^2}(-i(1-q^2)q^{2n+2}s)]=$$

$$=(1-q^2)\sum_{n=-\infty}^{\infty}q^{2n}\left[\overline{h(q^{2n})}\phi(q^{2n})E_{q^2}(i(1-q^2)q^{2n+2}s)+\overline{h(-q^{2n})}\phi(-q^{2n})E_{q^2}(-i(1-q^2)q^{2n+2}s)\right]=$$

$$=\int d_{q^2}z\overline{h(z)}\phi(z)\ {}_0\Phi_1(-;0;q^2(1-q^2)q^2zs).$$

Now let functional $r\in(S^{q^2})'$ be a q^2-singular one with the multiplicity p (see Definition 3.2), i.e.

$$r(s)=\sum_{k=0}^{p}\partial_s^k r_k(s),$$

where $r_k(s)$ are regular functionals. It follows from (3.12) that

$$(\mathcal{F}'_{q^2})^{-1}r = h(z) = \sum_{k=0}^{p}(iz)^k h_k(z). \tag{5.8}$$

We have from (3.7) and (5.5)

$$\mathcal{F}^{-1}_{q^2}(r*\psi)(s) = \mathcal{F}^{-1}_{q^2}\sum_{k=0}^{p}(-1)^k q^{-k(k+1)}\Lambda_s^{-k}\partial_s^k(r_k*\psi)(s) =$$

$$= \mathcal{F}^{-1}_{q^2}\sum_{k=0}^{p}(-1)^k(q^{-2}\Lambda_s^{-1}\partial_s)^k(r_k*\psi)(s) == \sum_{k=0}^{p}(iz)^k\overline{h_k(z)}\phi(z).$$

Then we obtain from (5.8)

$$\mathcal{F}^{-1}_{q^2}(r*\psi)(s) = \overline{h(z)}\phi(z).$$

■

Theorem 5.1 *Let $g \in (S^{q^2})'$ and $r(s)$ be a q^2-convolutor in S^{q^2}. Then the q^2-convolution $r*g$ (5.1) is the q^2-Fourier transform of the product of the q^2-distributions hf, where $h = (\mathcal{F}'_{q^2})^{-1}r$, and $f = (\mathcal{F}'_{q^2})^{-1}g$.*

Proof. For an arbitrary $\psi(s)$ and $\phi(z) = \mathcal{F}^{-1}_{q^2}\phi(z)$ we have from the definition of the q^2-distributions (5.1 and from the definition of the q^2-Fourier transform of the q^2-distributions 3.8

$$< r*g,\psi >=< g, r*\psi >=< f,\overline{h}\phi >=< fh,\phi > . \tag{5.9}$$

■

Corollary 5.1 *If $g(s)$ and $r(s)$ are q^2-convolutors in S^{q^2}, then them q^2-convolution is commutative one i.e. for an arbitrary $\psi(s) \in S^{q^2}$*

$$< r*g,\psi >=< g*r,\psi > . \tag{5.10}$$

The proof follows from commutativity of the product $f(z)h(z)$ in right side of (5.9). ■

Corollary 5.2 *If $g(s)$ is q^2-convolutor in S^{q^2}, then*

$$g*\delta_{q^2} = \delta_{q^2}*g = g. \tag{5.11}$$

Proof. $q^2-\delta$-function is the q^2-convolutor in S^{q^2} because it is the q^2-Fourier transform of $\frac{1}{2}$, which is the multiplicator in S_{q^2} (see [7]). It is follows from (4.10)

$$g*\delta_{q^2} = \int d_{q^2}\xi\delta_{q^2}(\xi)T^*_\xi g(s) = T^*_0 g(s) = g(s). \tag{5.12}$$

■

Proposition 5.4

$$\partial_s g*r = g*\partial_s r. \tag{5.13}$$

Proof. It is follows from (4.10)

$$\partial_\xi e_{q^2}(-(1-q^2)q^2\xi\partial_s) = -q^2 e_{q^2}(-(1-q^2)q^4\xi\partial_s)\partial_s.$$

Now (5.13) follows from (3.2) and (3.3).■

6 The q^2-pseudo differential operators

Consider the q^2-distribution $s_+^{\nu-1}$ [7]. For an arbitrary fixed $n \geq 0$

$$\int_0^\infty s^{\nu-1}\psi(s)d_{q^2}s = \int_0^1 s^{\nu-1}\Big[\psi(s) - \sum_{k=0}^n \frac{s^k}{k!}\psi^{(k)}(0)\Big]d_{q^2}s+$$

$$+\int_1^\infty s^{\nu-1}\psi(s)d_{q^2}s + (1-q^2)\sum_{k=0}^n \frac{1}{k!(1-q^{2(\nu+k)})}\psi^{(k)}(0).$$

It is seen from (3.2), that the last sum can be represented by the form

$$\sum_{k=0}^n \frac{(1-q^2)^{k+1}}{(q^2,q^2)_k(1-q^{2(\nu+k)})}\partial_s^k\psi(0).$$

On the other hand

$$< \delta_{q^2}, \partial_s^k\psi >= (-1)^k q^{k(k+1)} < \partial_s^k\delta_{q^2}, \psi > .$$

Hence $s_+^{\nu-1}$ is the meromorphic function of ν with the ordinary poles $\nu=-k, \;\; k=0,1,\dots$ with the residues

$$\mathrm{res}_{\nu=-k}\, s_+^{\nu-1} = (-1)^k q^{k(k+1)}\frac{(1-q^2)^{k+1}}{(q^2,q^2)_k}\partial_s^k\delta_{q^2}(s). \tag{6.1}$$

Consider now the q^2-Γ-function

$$\Gamma_{q^2}(\nu) = \frac{(q^2,q^2)_\infty}{(q^{2\nu},q^2)_\infty}(1-q^2)^{1-\nu}. \tag{6.2}$$

Obviously it is the meromorphic function with the ordinary poles $\nu=-k, \;\; k=0,1,\dots$ with the residues

$$\mathrm{res}_{\nu=-k}\,\Gamma_{q^2}(\nu) = \lim_{\nu\to-k}\frac{(q^2,q^2)_\infty(1-q^2)^{1-\nu}}{(1-q^{2\nu})\dots(1-q^{2\nu+2k-2})(q^{2\nu+2k+2},q^2)_\infty} =$$

$$= \frac{(1-q^2)^{k+1}}{(1-q^{-2\nu})\dots(1-q^{-2})} = (-1)^k q^{k(k+1)}\frac{(1-q^2)^{k+1}}{(q^2,q^2)_k}. \tag{6.3}$$

The next Proposition follows from (6.1) and (6.3).

Proposition 6.1 *The q^2-distribution $\frac{s_+^{\nu-1}}{\Gamma_{q^2}(\nu)}$ is the entire function of ν and*

$$\frac{s_+^{\nu-1}}{\Gamma_{q^2}(\nu)}\Big|_{\nu=-k} = \partial_s^k\delta_{q^2}(s), \quad k=0,\dots. \tag{6.4}$$

Definition 6.1 *For an arbitrary q^2-distribution $g \in (S^{q^2})'$ concentrated on the lattice $\{q^{2n}\}$ and absolutely q^2-integrable on any segment we will call the q^2-convolution*

$$g * \frac{s_+^{\nu-1}}{\Gamma_{q^2}(\nu)} \tag{6.5}$$

by the q^2-derivative of g $-\nu$ order if $\nu<0$ and by the q^2-primitive of g ν order if $\nu>0$.

In accordance with Definition 6.1 we introduce the designation

$$\partial_s^{-\nu} g = g * \frac{s_+^{\nu-1}}{\Gamma_{q^2}(\nu)}.$$

Moreover the q^2-derivative $-\nu$ order is the q^2-primitive ν order if $\nu > 0$.

For the proof of correctness of Definition 6.1 it is sufficient to check its correctness for $\nu = 0, \pm 1$ and to prove the following Proposition

Proposition 6.2 *For arbitrary ν and μ*

$$\frac{s_+^{\nu-1}}{\Gamma_{q^2}(\nu)} * \frac{s_+^{\mu-1}}{\Gamma_{q^2}(\mu)} = \frac{s_+^{\nu+\mu-1}}{\Gamma_{q^2}(\nu+\mu)}. \tag{6.6}$$

Let $\nu = 0$. It follows from (6.4) and (5.11)

$$g(s) * \frac{s_+^{\nu-1}}{\Gamma_{q^2}(\nu)}\Big|_{\nu=0} = (g * \delta_{q^2})(s) = g(s).$$

Let $\nu = -1$. It follows from (6.4), (5.11) and (5.13)

$$g(s) * \frac{s_+^{\nu-1}}{\Gamma_{q^2}(\nu)}\Big|_{\nu=-1} = (g * \partial_s \delta_{q^2})(s) = (\delta_{q^2} * \partial_s g)(s) = \partial_s g(s).$$

Finally let $\nu = 1$. As g is concentrated on the lattice $\{q^{2n}\}$ and

$$\left(T_\xi^* s^0\right)_+ = \begin{cases} 1 & \xi \le s \\ 0 & \xi > s, \end{cases}$$

then

$$g(s) * \frac{s_+^0}{\Gamma_{q^2}(1)} = \int_0^s d_{q^2}\xi g(\xi) = (1-q^2)s \sum_{m=0}^{\infty} q^{2m} g(q^{2m}s).$$

So the last function is the q^2-primitive of $g(s)$ because its q^2-derivative is $g(s)$.

Proof of Proposition 6.2. The statement of Proposition is trivial if ν and μ are integer. Let ν is not integer. Obviously

$$\partial_s^k s^{\nu-1} = (-1)^k q^{k(2\nu-k-1)} \frac{(q^{-2\nu+2}, q^2)_k}{(1-q^2)^k} s^{\nu-k-1}.$$

Using the properties of the q^2-binomial formula [10]

$$\sum_{k=0}^{\infty} \frac{(a, q^2)_k}{(q^2, q^2)_k} x^k = \frac{(ax, q^2)_\infty}{(x, q^2)_\infty},$$

and (4.10) for function $g(s)$ concentrated on the set $s \ge 0$ we obtain

$$\int_0^s d_{q^2}\xi g(\xi) T_\xi^* \frac{s^{\nu-1}}{\Gamma_{q^2}(\nu)} = \int_0^s d_{q^2}\xi g(\xi) \sum_{k=0}^{\infty} q^{2\nu k} \frac{(q^{-2\nu+2}, q^2)_k}{(q^2, q^2)_k} \frac{s^{\nu-k-1}}{\Gamma_{q^2}(\nu)} =$$

$$= (1-q^2)\frac{s^\nu}{\Gamma_{q^2}(\nu)}\sum_{m=0}^{\infty} q^{2m} g(q^{2m}s)\sum_{k=0}^{\infty}\frac{(q^{-2\nu+2},q^2)_k}{(q^2,q^2)_k}q^{2k(\nu+m)} =$$

$$= (1-q^2)\frac{s^\nu}{\Gamma_{q^2}(\nu)}\sum_{m=0}^{\infty}\frac{q^{2m}(q^{2m+2},q^2)_\infty}{(q^{2(\nu+m}},q^2)_\infty}g(q^{2m}s) =$$

$$= (1-q^2)\frac{s^\nu}{\Gamma_{q^2}(\nu)}\frac{(q^2,q^2)_\infty}{(q^{2\nu},q^2)_\infty}\sum_{m=0}^{\infty}\frac{q^{2m}(q^{2\nu},q^2)_m}{(q^2,q^2)_m}g(q^{2m}s) = (1-q^2)^\nu s^\nu \sum_{m=0}^{\infty}\frac{q^{2m}(q^{2\nu},q^2)_m}{(q^2,q^2)_m}g(q^{2m}s).$$

So

$$\partial_s^{-\nu} g(s) = (1-q^2)^\nu s^\nu \sum_{m=0}^{\infty}\frac{q^{2m}(q^{2\nu},q^2)_m}{(q^2,q^2)_m}g(q^{2m}s). \tag{6.7}$$

It is easily to prove by the induction on k that for any non integer ν and μ and for any integer $k \geq 0$

$$\sum_{l=0}^{k} q^{2\nu l}\left[\begin{array}{c} k \\ l \end{array}\right]_{q^2}(q^{2\mu},q^2)_l (q^{2\nu},q^2)_{k-l} = (q^{2\mu+2\nu},q^2)_k. \tag{6.8}$$

Consider now the composition of the operators $\partial_s^{-\nu}$ and $\partial_s^{-\mu}$ for non integer ν and μ. It follows from (6.7)

$$\partial_s^{-\mu}\partial_s^{-\nu} g(s) = (1-q^2)^\mu s^\mu \sum_{l=0}^{\infty}\frac{q^{2l}(q^{2\mu},q^2)_l}{(q^2,q^2)_l}(1-q^2)^\nu (q^{2l}s)^\nu \sum_{m=0}^{\infty}\frac{q^{2m}(q^{2\nu},q^2)_m}{(q^2,q^2)_m}g(q^{2m+2l}s) =$$

$$= (1-q^2)^{\nu+\mu}s^{\nu+\mu}\sum_{k=0}^{\infty}\frac{q^{2n}(q^{2\nu+2\mu},q^2)_k}{(q^2,q^2)_k}g(q^{2k}s)\sum_{l=0}^{k} q^{2\nu l}\left[\begin{array}{c} k \\ l \end{array}\right]_{q^2}\frac{(q^{2\mu},q^2)_l(q^{2\nu},q^2)_{k-l}}{(q^{2\mu+2\nu},q^2)_k} =$$

$$= (1-q^2)^{\nu+\mu}s^{\nu+\mu}\sum_{k=0}^{\infty}\frac{q^{2n}(q^{2\nu+2\mu},q^2)_k}{(q^2,q^2)_k}g(q^{2k}s) = \partial_s^{-(\nu+\mu)}g(s),$$

The last equality follows from (6.8). Proposition is proved. ■

In [7] the q^2-Fourier transforms of the q^2-distributions from the space S'_{q^2} z_+^ν and z_-^ν are calculated. It is follows from these formulas

$$\frac{s_+^{\nu-1}}{\Gamma_{q^2}(\nu)} = 2\Theta_0 \mathcal{F}'_{q^2}[A_\nu z_+^{-\nu} + \bar{A}_\nu z_-^{-\nu}], \tag{6.9}$$

where Θ_0 is determined by (2.4),

$$A_\nu = \frac{c_\nu (1-q^2)^{\nu-1}}{c_\nu^2 - \bar{c}_\nu^2}, \tag{6.10}$$

$$c_\nu = \sum_{m=-\infty}^{\infty}\frac{q^{-2\nu m} + i(1-q^2)q^{2(1-\nu)m}}{(1-q^2)^{-1}q^{-2m} + (1-q^2)q^{2m}}. \tag{6.11}$$

The next theorem is follows from Theorem 5.1 and (6.9)

Theorem 6.1 *(Addition theorem)*

$$A_\nu A_\mu = \frac{1}{2\Theta_0}A_{\nu+\mu}.$$

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